Analyzing Equilibrium Points in the Restricted Three-Body Problem with a Heterogeneous Primary and Albedo Effects

SHORT SYNOPSIS

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by

Keshav
Registration No.: MRUMADN012K220053

Under the supervision of

Dr. Advin Masih
Assistant Professor

Dr. Mohammad Javed Idrisi
Associate Professor
Department of Mathematics
College of Natural & Computational Science
Mizan-Tepi University, Ethiopia

Department of Sciences
(Mathematics)
School of Science

MANAV RACHNA UNIVERSITY
Sector-43, Aravali Hills, Faridabad

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Sector-43, Aravali Hills, Faridabad

Date: 26/07/24.

DECLARATION BY THE SCHOLAR

I hereby declare that the short synopsis entitled **"Analyzing Equilibrium Points in the Restricted Three-Body Problem with a Heterogeneous Primary and Albedo Effects"** being submitted in partial fulfillment of the requirement for the degree of Doctor of Philosophy in the Department of Sciences (Mathematics) under the School of Sciences of Manav Rachna University, Faridabad during the academic year 2024-2025.

It is the work that is expected to be carried out under the supervision of **Dr. Advin Masih** and co-supervision of **Dr. Mohammad Javed Idrisi.**

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Keshav

Registration Number MRUMADN012K220053

MANAV RACHNA UNIVERSITY

Sector-43, Aravali Hills, Faridabad

Date: 26/07/24

CERTIFICATE BY SUPERVISOR(S)

This is to certify that, the short synopsis entitled **"Analyzing Equilibrium Points in the Restricted Three-Body Problem with a Heterogeneous Primary and Albedo Effects"** is an original piece of work done by research scholar Keshav under the supervision of Dr. Advin Masih and co-supervision of Dr. Mohammad Javed Idrisi for the degree of Doctor of Philosophy in the Department of Sciences (Mathematics) under the School of Science of Manav Rachna University, Faridabad during the academic year 2024-2025.

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Dr. Advin Masih
Assistant Professor
Department of Sciences
Program: Mathematics
Manav Rachna University, Faridabad

Dr. Mohammad Javed Idrisi
Associate Professor
Department of Mathematics
College of Natural & Computational Science
Mizan-Tepi University, Ethiopia

ABSTRACT

This thesis proposal seeks to explore the influence of a heterogeneous primary, characterized by non-uniform density distribution, and the inclusion of albedo effects on equilibrium points within the Restricted Three-Body Problem (RTBP). By employing analytical methods and computational simulations, the purpose of the study is to clarify how such variables impact the stability and location of equilibrium points and other equilibrium configurations in the dynamical system. Understanding these variation can provide important information about celestial mechanics, with potential applications in spacecraft trajectory planning, satellite mission design, and our comprehension of celestial body interactions and evolution.

Keywords: RTBP, equilibrium points, albedo effects, Stability analysis, Heterogeneous primary sources, Radiation effects.

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Chapter 1: Introduction

The restricted three-body problem concerns the motion of a small mass in three dimensional or in a single plane that is affected by the gravitational fields of two massive bodies. The primary bodies are rotating around their center of mass in either an elliptical or circular orbit due to their mutual gravitation. If the primary bodies follow elliptical path around their center of mass, it is called elliptic restricted three-body problem (ER3BP or ERTBP). On the other hand, if the primary bodies have a circular orbit, it is called circular restricted three-body problem (CR3BP, CRTBP, RTBP, or R3BP). Within this problem, there are five equilibrium points. Three of are collinear, while the other two are not. For mass ratios $\mu \leq 0.038520896505$, the non-collinear equilibrium points are stable as they maintain their position. However, the collinear equilibrium points are unstable. Szebehely (1967).

1.1 Mathematical Background

Let m_1, m_2, and m_3 ($m_1 \geq m_2$) be the three masses. Let the mass m_1 and m_2 which are point masses or spheres rotate around their center of mass O in a circular orbit at a fixed angular velocity ω. Let m_3 be the smallest mass is moving in m_1 and m_2's plane of motion, influencing but not changing their motion.

Visualize a situation where two masses m_1 and m_2, are connected by a line along the X-axis, with the origin at point O. Inside the motion's plane of m_1 and m_2, the Y-axis is perpendicular to the X-axis through O. Now, let's introduce a synodic coordinate system denoted as $O\ (xyz)$. This system rotates around the Z-axis at an angular velocity of ω. Initially, it conforms to initial framework $O\ (XYZ)$, where the Z-axis coincides with the z-axis.

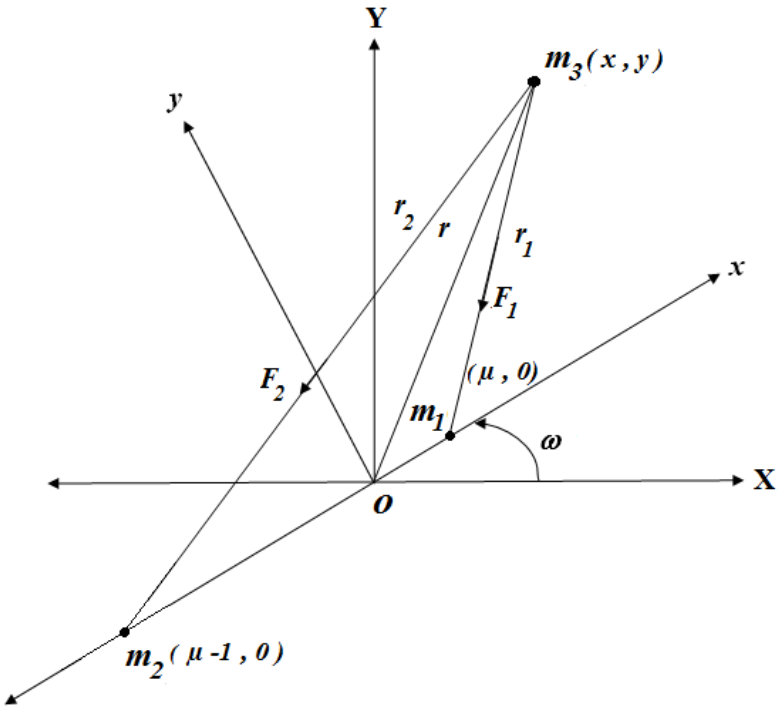

Fig. 1: The Configuration of the Restricted Three-Body Problem

To find the vector equation of motion in a synodic frame is, (McCusky, 1963)

$$m_3\left\{\frac{\partial^2\vec{r}}{\partial t^2}+2\vec{\omega}\times\frac{\partial\vec{r}}{\partial t}+\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)\right\}=\vec{F}_1+\vec{F}_2, \tag{1}$$

where

$$\vec{r}=x\,\hat{i}+y\,j,$$

$$\vec{\omega}=n\hat{k},$$

$$\frac{\partial^2\vec{r}}{\partial t^2}=\ddot{x}\,\hat{i}+\ddot{y}\,j,$$

$$r_1^2=\left(x-x_1\right)^2+y^2,$$

$$r_2^2=\left(x-x_2\right)^2+y^2,$$

$\vec{F}_1=$ Gravitational force on m_3 due to m_1.

$\vec{F}_2=$ Gravitational force on m_3 due to m_2.

The motion in Cartesian form is given by

$$\ddot{x} - 2n\dot{y} - n^2 x = \frac{Gm_1(x_1 - x)}{r_1^3} + \frac{Gm_2(x_2 - x)}{r_2^3}, \tag{2}$$

$$\ddot{y} + 2n\dot{x} - n^2 y = \frac{Gm_1(-y)}{r_1^3} + \frac{Gm_2(-y)}{r_2^3}. \tag{3}$$

using the terminology and notation of Szebehely (1967), the primaries masses sum is assumed to be unity, sapration between the primaries is also unity and gravitational constant per unit time is equal.

i. e. $m_1 + m_2 = 1,\ d(m_1, m_2) = 1,\ G = 1.$

Let $\mu = \dfrac{m_2}{m_1 + m_2} \Rightarrow m_2 = \mu$ and $m_1 = 1 - \mu.$

in the moving coordinate system, the equation governing the motion of infinitesimal mass m_3 can be expressed in a dimensionless form. (Szebehely, 1967)

$$\ddot{x} - 2n\dot{y} = x - \frac{(1-\mu)}{r_1^3}(x - \mu) - \frac{\mu}{r_2^3}(x + 1 - \mu), \tag{4}$$

$$\ddot{y} + 2n\dot{x} = y\left[1 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3}\right]. \tag{5}$$

where

$$r_1^2 = (x - \mu)^2 + y^2,$$

$$r_2^2 = (x + 1 - \mu)^2 + y^2,$$

$$n = 1.$$

Now we define a function Ω, such that

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}. \tag{6}$$

Hence the Equations (4) and (5) become

$$\ddot{x} - 2n\dot{y} = \Omega_x, \tag{7}$$

$$\ddot{y} + 2n\dot{x} = \Omega_y. \tag{8}$$

The Jacobi integral is (9)

$$(\dot{x}^2 + \dot{y}^2) = v^2 = 2\Omega - C,$$

where v is the velocity of the small mass m_3.

1.2 Equilibrium solution's stability

A dynamical system's stability characterizes how it responds to perturbations. When a particle is slightly moved from its balanced position and stays close to the balanced point after the disturbance, the balanced point is considered stable.

(i) Equilibrium Solution:

Consider an ordinary differential equation,

$$\dot{x} = X(x),$$

where

$$x = (x_1, x_2, x_3, \ldots, x_n)$$

and

$$X = (X_1, X_2, X_3, \ldots, X_n).$$

We suppose that classic existence theorems, such as Cauchy-Lipschitz, are satisfied.

If the path in the X-space is represented by a single point 'a' where $x(t_0) = a$, then the solution x(t) is considered an equilibrium solution. An equilibrium point, also referred to as a stationary or critical point, is the result $\dot{x} = 0$ of the equation X(x) = 0 (Moulton, 1914).

(ii) Stable equilibrium solution:

When there is a positive number ε, there is also a positive number δ(ε) such that the equilibrium point a or the solution X = a is stable and satisfies the perturbation condition,

for all $t > t_0$,

$$| x(t) - a | < \varepsilon,$$

In any other case of x = a, the equilibrium solution is unstable.

(iii) Asymptotically stable solution:

The solution is referred to as asymptotically stable as it approaches the equilibrium point as $t \to \infty$ the solution is asymptotically stable with the condition is

$$\lim_{t \to \infty} x(t) = a.$$

(iv) Completely stable solution:

The equilibrium solution is considered completely stable when it is asymptotically stable over a large domain or within the domain where these initial conditions exhibit asymptotic stability.

1.3 The stability of linear system theorem

Consider an independent system of differential equations

$$\dot{x}_i = X(x_1, x_2, ..., x_n), i = 1, 2, ..., n$$

having $x_i = 0$ an equilibrium point.

Suppose the system of equations can be in the form

$$\dot{x}_i = \sum_{j=1}^{n} p_{ij}\, x_j + X_i'(x_1, x_2, ..., x_n). \tag{10}$$

Where p_{ij} are constants and $X_i'(x_1, x_2, ..., x_n)$ are power series in x_1, x_2... x_n that start with terms of at least second degree. Then the system of equation

$$\dot{x}_i = \sum_{j=1}^{n} p_{ij}\, x_j \tag{11}$$

is called a linear system. We refer to the system in Equation (10) as a non-linear system. The following properties of the characteristic roots of the matrix A = (pij) determine the stability of the linear system (11) (S.L. Ross, 1974):

1. When characteristic roots of the equation have negative real no. the equilibrium points are said to be asymptotically stable which is also true for several cases.
2. Equilibrium points are unstable if characteristic roots possess real positive numbers. This fact holds true across numerous scenarios.
3. When some specific roots have positive real components, it affects the stability of equilibrium points. If the mixed secular and periodic terms are present in these solutions, then the equilibrium points unstable.
4. When the roots are negative real no., the solution is stable. The solution is unstable if there is a positive root. For numerous roots, these propositions also hold true.

Chapter 2: Literature Review

Famous mathematicians like as Euler and Lagrange (1772), Jacobi (1836), Hill (1878), Poincare (1899), Levi-Civita (1905), and Birkhoff (1915) have made significant contributions to this challenging topic. A synodic (rotating) coordinate system was initially presented by Euler in 1772. Subsequently, Jacobi (1836) developed the Jacobi integral of motion which is generated independently in this coordinate system. Hill (1878) demonstrated that the Earth-Moon distance model is always constrained from above which is assuming accurate for the Sun-Earth-Moon system. Brown (1896) has gave a lunar theory at that time.

In 1899, Poincaré released Méthodes Nouvelles, which focused on qualitative aspects of celestial mechanics and new ideas like phase space surfaces of section. These qualitative techniques were expanded by Birkhoff (1915). Many academicians have delved deeply into the essential topic of regularization, including Thiele (1892), Painlevé (1897), Levi-Civita (1903), Burrau (1906), Sundman (1912), and Birkhoff (1915).It was demonstrated by Painleve that, for n = 3, all singularities are collisions. Sundman solved the restricted three-body issue on the full plane (after singularities are eliminated by regularization) by discovering the involvement of well-known function of uniformly convergent infinite series. It can be concluded that the restricted problem of three bodies has been fully solved because such global regularizations exist for it. However, this technique is not very useful because it doesn't deal with stability difficulties, permitted zones of motion, and other issues. Moreover, an excessively high quantity of terms from Sundman's series must be used in order to achieve any level of accuracy close to that needed for astronomical observations.

Szebehely (1967) explored the impact of small Coriolis disturbances on uniformity of balance positions in the restricted three-body problem. The research provided a linear correlation that could be applied to both linear and non-linear situations with critical mass parameter to the degree of Coriolis force deviation.

Zuzaniedzielska (1994) study examined how first- and second-order resonances impact the stability of triangle equilibrium points. The findings reveal that while second-order resonances have the potential to produce instability, first-order resonances do not.

The study maps out areas of instability within a two-dimensional parameter space.

Cabral H.E. and Xia Z. (1995) studied the subharmonic splits in the r3bp. Their work provides a detailed bifurcation analysis near circular solutions when one of the primary bodies is small, using Melnikov integrals to examine subharmonic solutions.

Khanna M. and Bhatnagar K.B. (1999) discovered a triaxial body with a spherical oblique function. It had a larger primary body and a smaller one. The study identified five equilibrium points - two non-collinear and three collinear. The non-collinear positions were stable throughout the range $0 \leq \mu < \mu c$ (critical mass parameter), while the collinear points were found to be unstable. The research also suggests that the triangular points might be located within the same mass parameter range, but the elliptical orbits with varying lengths of time.

Viswanath D. (2005) focused on linear stability analysis to understand the resonant periodic motion in the RTBP. The paper discusses the equations governing the problem, highlighting the orbits for a massless particle when the mass ratio μ is zero. It then explores how these orbits change to periodic motions when μ becomes greater than zero. The paper provides expressions for the characteristic multipliers and analyzes the types of periodic motions observed as μ approaches zero for typical values of eccentricity and rational periods.

Raheem R. and Singh J. (2006) examined the stability of the balance point when minor disruptions caused by centrifugal and Coriolis forces, as well as the effects of radiation pressure from the primary and oblateness, were present. They found that instability occurs at collinear positions. Additionally, they showed that the non-collinear points remain stable for $0 \leq \mu < \mu_c$, but become unstable for $\mu_c \leq \mu \leq 1/2$, where μ is critical mass parameter. The Coriolis force contributes to stabilization, but the primary oblateness, radiation, and centrifugal force have destabilizing effects. Consequently, the stability range of the triangular points is reduced.

Razaq et.al. (2006) examined various factors that influence the stability of balance points in a celestial system. Their analysis considered radiation pressures, the oblong shape of celestial bodies, Centripetal and centrifugal forces, and minor disturbances.
The findings indicate that collinear points remain unstable. However, the triangular points exhibit stability within a certain range of the critical mass parameter μ. specifically, they are stable when $0 \leq \mu < \mu_c$, but become unstable for $\mu_c \leq \mu \leq 1/2$.Interestingly, the Coriolis force contributes to stabilizing the triangular points. But this stabilizing effect is counteracted by destabilizing influences from the primary oblateness, radiation, and centrifugal forces. Overall, the existence of these destabilizing factors narrows the stability range for non-collinear equilibrium points.

Singh B. (2007) explored the non-linear stability of equilibrium points in the R3BP. The focus was on the generalized photo-gravitational scenario involving Poynting-Robertson drag. The study involved normalizing the first and subsequent orders of Hamiltonian and applying the KAM theorem to assess non-linear stability conditions. The investigation identified three critical mass ratios. Interestingly, the triangular equilibrium points display consistent behavior in a non-linear sense, except at these critical mass ratios where the KAM theorem is inapplicable.

Singh B. and Kushvah (2008) explored the relationship between radiation pressure and the equilibrium locations in the modified photogravational restriction of three-body system. The massive primary is treated as a radiation source, and the smaller primary as an irregular polygon. The study discusses the Poynting-Robertson effect caused by radiation pressure, noting that the non-collinear points are asymmetrical, and the linear points deviate from the axis connecting the two main bodies. By Poynting–Robertson effect, all these points are unstable in a linear sense.

Singh J. (2009) looked at how changes in the three-body problem with different masses affect the nonlinear stability of triangle points. It found that, except for three specific mass ratios, the triangle positions are stable for all mass ratios within the linear stability range when small changes in the Centripetal and centrifugal forces are considered.

Douskos, C.N. (2010) investigated the collinear equilibrium points of Hill's radiation and oblateness problem. They suggested that fractals could be areas of interest for Newton's approach. The study looked at the presence, positions, and stability of these collinear e-quilibrium points under different parameter scenarios.

Idrisi M.J. and Taqvi Z.A. (2013) examined the effects of oblateness, radiation, and minor disturbances on the stability of equilibrium positions in the r3bp. They derived equations of motion using elliptic integrals for the infinitesimal mass and analyzed the stability of both collinear and non-collinear balanced locations. Their findings indicated that the collinear points were generally unstable, while the triangular points were stable within certain ranges of the mass parameter.

Idrisi M.J. and Taqvi Z.A. (2013) focused on a specific version of the three-body problem where one of the main objects is a spheroid. The study looked at five collinear equilibrium points and some off-center locations on the unit circle around the larger primary body. The researchers then derived the equations of motion, which involved elliptic integrals. Similar to their previous work, they found that all the observed equilibrium points were unstable.

Idrisi M.J. (2014) examine the presence and consistency of balance positions in the restricted three-body problem with an spheroid as the smaller primary. The study found five unstable non-collinear points as well as collinear equilibrium positions. Three collinear and non-collinear balance positions were identified within the unit circle.

Idrisi M.J. and Amjad M. (2015) studied the r3bp, examining the Earth as an ellipsoid. The equilibrium points they identified can be classified as collinear and non-collinear, falling within the range of 0 to 45^{o}. These points form a right triangle, with three collinear and non-collinear points. Notably, all observed balance points in this range display instability.

Narayan A. et.al. (2015) examined how radiation and triaxiality affect the triangular e-quilibrium positions in the elliptical restricted three-body problem. They assumed both primary

bodies were triaxial and emitting radiation. The research observed stable motion around these points under specific conditions related to radiation pressure and oblate triaxiality. The critical mass ratio was found to depend on several parameters, and the effects became destabilizing as these parameters increased.

Shalini et.al. (2016) studied the nature of non-collinear balance point L_4 in the restricted three-body problem. They examined a scenario where the smaller primary is a three-layered heterogeneous spheroid with different densities. Moser's theorem doesn't apply at three specific mass ratios, μ_1', μ_2', and μ_3', where the triangular equilibrium point remains stable throughout the linear stability range of $0 < \mu < \mu_c$.

Idrisi M.J. (2017) informed that the Sun is one of the main sources of radiation in our solar system. Some of the radiation that strikes other planets or asteroids is absorbed, while some is reflected back into space - this is called albedo. Both albedo and direct radiation from the Sun have an impact on spacecraft. In this work, the author examined how an ellipsoid-shaped primary body with lower mass can affect the stability and equilibrium positions in a restricted three-body problem. Five equilibrium positions were found, two of which are triangular and others are collinear. The triangular points are stable when a critical mass parameter μ is less than a certain value μ_c. However, the linear points remain unstable. The author provided a real-world application using the example of the Earth-Moon system.

Idrisi M.J. (2017) investigated the disturbed equilibrium locations within the RTBP. They modeled the smaller primary body like an oblate spheroid. The analysis uncovered both collinear and non-collinear balance points, all of which were found to be unstable. However, the three collinear equilibrium positions exhibited stability, but only for a critical value of the mass parameter $\mu \leq \mu_c$, which is determined by the expression $\mu_c = \mu_o$ - (0.00891747 + 0.222579k)α, where μ_o represents the parameter of critical mass in the classical case.

Ansari et.al. (2018) examined two main bodies as irregular oblate spheres with three layers of varying densities. They also introduced a compact body with variable mass. Their investigation involved mapping equilibrium positions in different planes (within the plane or outside it),

analyzing zero velocity curves, identifying areas of feasible motion, and examining surfaces (including the zero velocity surface with projection and the Poincaré surface of section). They assessed the set of equations using Jeans' Law. Finally, the researchers applied the Meshcherskii space-time inverse transformation to assess the stability of the balanced points, finding that all of these points are unstable.

Idrisi M. and Ullah M.S. (2018) studied the relationship between reflected and direct radiations. They showed how the oblateness of the compact primary affected the triangular balance positions in the elliptic restricted three-body problem. When the radiation factor α=0 and the oblateness factor σ=0, the equilibrium points $L_{4,5}$ form an isosceles triangle with the primaries. As the eccentricity of the primaries (er) increases, the equilibrium points $L_{4,5}$ move downward vertically. When α=0 but $\sigma \neq 0$, the equilibrium shifts to the right from its initial position, forming a scalene triangle with the primaries. The equilibrium points $L_{4,5}$ also decrease vertically as er increases. If both σ and α are non-zero, then the equilibrium positions $L_{4,5}$ and the primaries form a scalene triangle. As the value e (eccentricity) increases, the $L_{4,5}$ Lagrangian points shift leftward and downward. The stability of relative equilibrium points was analyzed, taking into account the oblate shape of the smaller primary body. This revealed different configurations and stability conditions for the system.

Idrisi et.al. (2020) looked at how albedo impacts equilibrium positions. They studied it with the elliptic restricted three-body problem. The researchers found three points lined up and two not lined up by combining orbital shape effects and albedo. The points not in a line were stable for some mass values less than a limit. However, the lined-up points did not stay stable for any mass values.

Mariusz E. Grotte and Marcus J. Holzinger (2020) looked at how sunlight reflects off sails. This reflection creates a type of pressure, called albedo radiation pressure. Their study added albedo effects to the math describing solar sail motions. The math showed more points where solar sails could stay still between two key locations, L_1 and L_2. However, these stillpoints were often unstable. The scientists said adjusting the sail's angle could make some stillpoints stable over time. But keeping sails still at those spots would be hard.

Singh J. and Haruna S. (2020) looked at balance points and stability in the photogravitational restricted three-body problem with oblateness. They studied a spheroid as the smaller object. The bigger object radiated mass. The smaller object had three layers of differing densities. They found two stable triangular areas when considering radiation pressure, diversity, and obliquity as mass parameters. There were five collinear equilibrium points. All were unstable due to the smaller object's asymmetry, the bigger object's radiation, the smaller object's obliquity, and certain mass parameter values.

Singh J. and Haruna S. (2020) analysis tackled the three-body elliptic issue. They looked at how albedo and orbital eccentricity affect equilibrium positions. Three collinear and two non-collinear equilibrium points were discovered. The collinear points proved unstable regardless of mass parameter values μ. In contrast, the triangular points showed stability but only at a specific μ value.

Idrisi M.J. and Ullah M.S. (2021) determined non-collinear equilibrium positions in the circular restricted three-body photogravitational problem with Stokes drag and radiation pressure. The study identified two asymptotically stable non-collinear equilibrium position in a specific interval $0.6149 \leq q \leq 1$ for the mass parameter $\mu = 0.01$.

Ansari et.al. (2021) looked into the problem of generalized CR3BP. It involves a finite straight region with a primary and secondary that are not the same. The study checked if stationary points exist and if they are stable. It also looked at having a primary with N layers, radiating finite straight segments, and Coriolis and centrifugal forces within the CR3BP framework.

Idrisi M.J. and Ullah M.S. (2021) examined how albedo affects the elliptical three-body issue. They studied out-of-plane equilibria. The primary model was flattened. Numerical and graphical computations probed parameters μ, α, e, k, σ, μ, α, e, k, σ represent mass parameter. These were linearly unstable.

Singh et.al. (2023) explored the restricted three-body problem's unstable nature. They studied the triangular points L_4, L_5. They considered effects of uneven main bodies, a straight line smaller body. They derived conditions considering Coriolis, Centrifugal forces. They normalized Hamiltonian to Birkhoff's normal form, crucial for studying instability. They checked KAM Theorem conditions. For mass ratios μ_1, μ_2, μ_3, instability was seen. However, for mass ratio range $0<\mu<\mu_c$, L_4 and L_5 stayed stable within linear bounds.

Kumari et.al. (2023) conducted a study to examine the movement of a small mass m in a restricted three-body system. They used a non-homogeneous spheroid with three layers in which less enormous primary m_2, while the more enormous primary m_1, and was treated as a point mass. The researchers derived the motion equations and the potential of the proposed model. The study identified five equilibrium positions for mass parameter μ. The collinear points were found to be unstable, while the non-collinear positions L_4 and L_5 were conditionally stable for μ in the range $(0, \mu_c)$, where $\mu_c = 0.58688k_1$.

Kumari et.al. (2024) explored the motion of an infinitesimal mass m_3 within the context of restricted three-body problem. They analyzed primary and heterogeneous three-dimensional rigid bodies with N layers. The researchers identified five equilibrium points denoted as Li (i = 1, 2, 3, 4, 5) in their proposed model. These equilibrium positions include three collinear points and two non-collinear points. While the collinear positions are stable, the non-collinear positions remain stable only for mass criterion $(0, \mu_c)$, where μ_c is calculated as $0.25681k_1 - 1.31619k_2 - 2.24318k_3 - 6.10258k_4$

Vincent et.al. (2024) studied the motion of an infinitesimal mass moved with the Newtonian gravitational forces of two larger objects, one an oblate and the other an elongated asteroid, resulting in a spinning polarization with two equal masses. In this paper, they had found the perturbing forces of existence, positions, and the equilibrium point’s linear stability. They enabled the areas of motion specified by the isoenergetic curves and zero-velocity surface, also the numerical determination of equilibrium locations utilizing the binary system's motion Luhmann-16.

Chapter 3: Description of Broad Area

Description

The proposed research investigates the equilibrium point's behavior in terms of RTBP, a classical model in astronomical dynamics. Specifically, the study focuses on how the presence of a heterogeneous primary body, characterized by non-uniform density distribution, and the consideration of albedo effects impact the stability and location of equilibrium points within the system. By employing mathematical analysis, numerical simulations, and computational techniques, the research aims to deepen our understanding of multi-body dynamics in space and their implications for spacecraft trajectory planning, satellite mission design, and the study of celestial body interactions.

Broad Area

The research falls within the interdisciplinary domain of Astrodynamics, Celestial Mechanics, and Dynamical Systems theory. It integrates principles from Physics, Mathematics, and Computational Science to investigate the behavior of celestial objects in multi-body systems. The study's focus on equilibrium points in the RTBP with a heterogeneous primary and albedo effects contributes to advancements in fundamental celestial mechanics while also offering practical insights for space mission planning and exploration.

Chapter 4: Objectives of the study

The study has the following objectives:

- Investigate how a Heterogeneous primary body affects the stability, position and dynamics of balance points and other configuration in terms of R3BP.
- Utilize mathematical analysis and computational simulations to examine the equilibrium points, their stability criterion and behavior under varying condition of Heterogeneous body and albedo effect in the modified RTBP.
- Compare and contrast the dynamical behavior of equilibrium points between the classical RTBP and the modified model with a heterogeneous primary and albedo effects.
- Provide insights into the broader implications of the research for Astrodynamics, Celestial Mechanics, and the interdisciplinary field of Dynamical Systems theory.

Chapter 5: Methodology

The methodology of the proposed thesis involves a combination of analytical techniques and computational simulations to investigate how equilibrium points behave within the framework for the problem with restricted three bodies (RTBP) with a heterogeneous primary and albedo effects. The methodology can be outlined as follows:

1. **Literature Review:** Conduct a comprehensive review of existing literature on the RTBP, celestial mechanics, and related studies investigating stability criterion for equilibrium locations. This will establish a foundation for understanding the theoretical background and previous research in the field.

2. **Formulation of the Model:** Develop a mathematical model that incorporates the effects of a heterogeneous primary body and albedo on the dynamics of the RTBP. This involves defining the governing equations of motion, considering gravitational interactions, and incorporating parameters related to non-uniform density distributions and albedo properties.

3. **Analytical Analysis:** Utilize mathematical techniques such as perturbation theory, Hamiltonian mechanics, and stability analysis to obtain equilibrium point's analytical expression and their stability conditions in the modified RTBP model. This systematic techniques provide understanding of how equilibrium points behave in idealized scenarios.

4. **Numerical Simulations:** Improve computational simulations by applying numerical techniques like numerical integration schemes (e.g., Runge-Kutta methods) to resolve the equations of motion for the RTBP with the heterogeneous primary and albedo effects. Simulations will be conducted for various scenarios, including different configurations of primary bodies, albedo values, and initial conditions.

5. **Data Analysis:** Analyze the results obtained from both the analytical and numerical approaches to examine how equilibrium points behave in the modified RTBP model with that of the classical RTBP. Evaluate how the presence of a heterogeneous primary and albedo effects influence the stability and location of equilibrium points.

6. **Validation:** Verify the outcomes of the computational simulations through their comparison with analytical forecasts and existing literature where applicable. This ensures the accuracy and reliability of the findings obtained through numerical methods. For computational simulation, we will use the software **MATHEMATICA**.

7. **Interpretation and Conclusion**: Interpret the findings under the framework of the research objectives and broader implications for Astrodynamics, Celestial Mechanics, and Space Mission Planning. Draw conclusions regarding the result of a heterogeneous primary and albedo effects on equilibrium points in the RTBP and explore various future research directions.

By following this methodology, the proposed thesis aims to give a comparative study of equilibrium locations in the RTBP with a heterogeneous primary and albedo effects, contributing to advancements in celestial mechanics and space exploration.

Chapter 6: Expected outcomes of the Research

The expected outcomes of this research are as follows:

- **Characterization of Equilibrium Points**: A detailed comprehension of how albedo factors impact the stability and location of equilibrium positions in the RTBP, as well as the presence of a heterogeneous primary body.
- **Identification of Stability Criteria:** Determination of stability criteria for equilibrium points in the modified RTBP model, considering factors such as non-uniform density distributions and albedo properties of celestial bodies.
- **Insights into Dynamical Behavior:** Insights into the dynamical behavior of equilibrium points and other equilibrium configurations under varying conditions of heterogeneous primary bodies and albedo effects, obtained through both analytical and numerical analyses.
- **Comparison with Classical RTBP:** Comparison of how equilibrium points behave the modified RTBP model and the classical RTBP, highlighting the effects of introducing a heterogeneous primary and albedo considerations.
- **Practical Implications:** Identification of practical implications for spacecraft trajectory planning, satellite mission design, and celestial body exploration based on the findings regarding the balance of equilibrium points within the modified RTBP model.
- **Contribution to Celestial Mechanics:** Contribution to advancements in celestial mechanics by expanding the understanding of multi-body dynamics in space and providing insights into the effects of non-uniform density distributions and albedo properties on the behavior of celestial objects.
- **Potential for Further Research:** Identification of possible direction for additional investigation such as exploring additional complexities or extensions to the model, investigating real-world applications, and addressing remaining questions or uncertainties arising from the study.

Overall, the research is expected to understanding the equilibrium point dynamics in the RTBP with primary and albedo effects that are diverse, with significance for both theoretical celestial mechanics and practical space mission planning and exploration.

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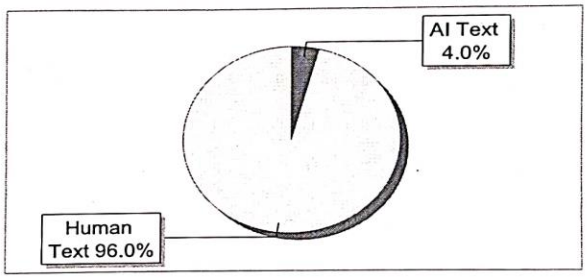

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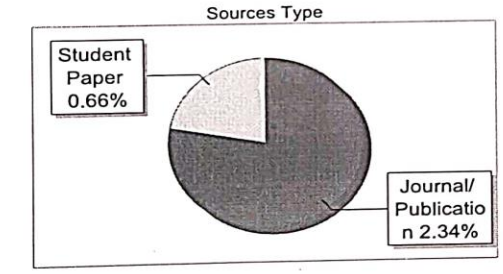

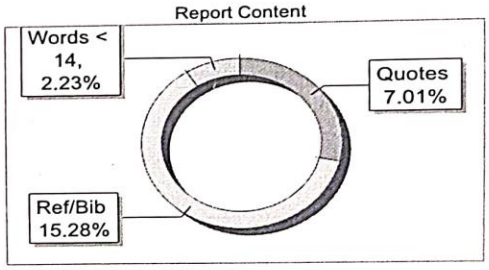

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